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A UNIFIED THEORY FOR REAL VS. COMPLEX RATIONAL CHEBYSHEV APPROXIMATION ON AN INTERVAL

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ABSTRACT. A unified approach is presented for determining all the constants $\gamma_{m,n}$ $(m \geq 0, n \geq 0)$ which occur in the study of real vs. complex rational Chebyshev approximation on an interval. In particular, it is shown that $\gamma_{m,m+2} = 1/3$ $(m \geq 0)$, a problem which had remained open.

1. INTRODUCTION

Let π_m^r and π_m^c denote, respectively, the sets of polynomials of degree at most m, with real and complex coefficients. For any pair (m, n) of nonnegative integers, $\pi_{m,n}^r$ denotes the set of rational functions of the form $p(x)/q(x)$, where $p \in \pi_m^r$ and $q \in \pi_n^r$, and we define $\pi_{m,n}^c$ analogously as the set of rational functions of the form $p(x)/q(x)$ where $p \in \pi_m^c$ and $q \in \pi_n^c$. Let $\|\cdot\|_I$ denote the supremum norm on $[-1, 1]$, i.e., $\|f\|_I := \sup_{x\in[-1,1]}|f(x)|$. If $C^r[-1, 1]$ denotes the set of all continuous real-valued functions on $[-1, 1]$, then, for f in $C^r[-1, 1]$, we set

$$(1.1)\qquad \begin{aligned} E_{m,n}^r(f) &:= \inf\{\|f-g\| : g \in \pi_{m,n}^r\}, \\ E_{m,n}^c(f) &:= \inf\{\|f-g\| : g \in \pi_{m,n}^c\}. \end{aligned}$$

For $f \in C^r[-1, 1]$, it is well known that there exist functions $h \in \pi_{m,n}^r$ and $g \in \pi_{m,n}^c$ satisfying $E_{m,n}^r(f) = \|f-h\|_I$ and $E_{m,n}^c(f) = \|f-g\|_I$. In fact, h can be characterized by the length of the alternation set of $f-h$ (cf. Meinardus [2, p. 162]). Less is known about the g for which $E_{m,n}^c(f) = \|f-g\|_I$. Since $\pi_{m,n}^r \subseteq \pi_{m,n}^c$, then evidently $E_{m,n}^c(f) \leq E_{m,n}^r(f)$, but it is not obvious that strict inequality can hold. What is surprizing here is that, for each $m \geq 0$ and $n \geq 1$, there is a *real* continuous function f on the *real* interval $[-1, +1]$ for which

$$(1.2)\qquad E_{m,n}^c(f)/E_{m,n}^r(f) < 1.$$

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(For a recent treatment of this, which covers the early contributions of A. A. Gončar, K. N. Lungu, and Saff and Varga, see [6, Chapter 5].)

Saff and Varga [4] raised the question as to how *small* the ratio $E^c_{m,n}(f)/E^r_{m,n}(f)$ can be for a fixed integer pair (m, n). More precisely, they asked which values the numbers $\gamma_{m,n}$ take on, where

$$\gamma_{m,n} := \inf\{E^c_{m,n}(f)/E^r_{m,n}(f) : f \in C^r[-1, 1]\backslash\pi^r_{m,n}\}. \tag{1.3}$$

Three recent papers have described the behavior of $\gamma_{m,n}$ in terms of (m, n). First, Trefethen and Gutknecht [5] established, by means of a direct construction, the surprising result that

$$\gamma_{m,n} = 0, \text{ for each pair } (m, n) \text{ of nonnegative integers with } n \geq m + 3. \tag{1.4}$$

Next, Levin [1] established the complementary result that

$$\gamma_{m,n} = 1/2, \text{ for each pair } (m, n) \text{ of nonnegative integers with } m + 1 \geq n \geq 1. \tag{1.5}$$

Levin's proof of (1.5) consisted of a direct construction to show that $\gamma_{m,n} \leq \frac{1}{2}$, and an algebraic method to show that $\gamma_{m,n} < \frac{1}{2}$ was impossible for $m + 1 \geq n \geq 1$. The results of (1.4) and (1.5) leave open only the case $\gamma_{m,m+2}$ $(m \geq 0)$. For this case, Ruttan and Varga [3], also by means of a direct construction, have more recently shown that

$$\gamma_{m,m+2} \leq 1/3 \qquad (m \geq 0). \tag{1.6}$$

This result, however, leaves open the question of the actual values of $\gamma_{m,m+2}$, $m \geq 0$, allowing speculation that perhaps $\gamma_{m,m+2}$ might be zero or even that $\gamma_{m,m+2}$ might take on different values as m varies.

Our object here is to complete this topic by showing that

$$\gamma_{m,m+2} = 1/3 \qquad (m \geq 0). \tag{1.7}$$

In the process of establishing (1.7), we develop two results for general complex rational functions which provide a *unified* approach to the problem of determining the values of $\gamma_{m,n}$.

2. Upper bounds for $\gamma_{m,n}$

Table 1 lists the values of $\gamma_{m,n}$ established in [5] $(n \geq m + 3)$ and in [1] $(1 \leq n \leq m+1)$, together with the values of $\gamma_{m,n}$ $(n = m+2)$ which follow from [3] and the results to be developed below. Evidently, $\gamma_{m,n}$ takes on only four distinct values: $0, 1/3, 1/2,$ and 1. The value 1 occurs only when $n = 0$ and is a consequence of the well known fact that the best uniform approximant, from $\pi^c_{m,0}$, of any real-valued continuous function on $[-1, 1]$ is a *real* polynomial, whence $E^r_{m,n}(f) = E^r_{m,n}(f)$. The remaining values $0, 1/3,$ and $1/2$ occur in

	0	1	2	3	4	5
0	1	1	1	1	...	...
1	1/2	1/2	1/2	1/2	...	...
2	1/3	1/2	1/2	1/2	⋱	⋱
3	0	1/3	1/2	1/2	⋱	⋱
4	0	0	1/3	1/2	⋱	⋱
5	0	0	0	1/3	⋱	⋱
6	0	0	0	0	⋱	⋱
7	⋮	⋮	⋮	⋮	⋱	⋱

TABLE 1. Values of $\gamma_{m,n}$ $(m \geq 0; n \geq 0)$

the regions $R_1 := \{(m, n) : n \geq m+3\}$, $R_2 := \{(m, n) : n = m+2\}$, and $R_3 := \{(m, n) : 1 \leq n \leq m+1\}$, respectively, of Table 1.

In establishing the sharp upper bounds for $\gamma_{m,n}$ for a given region R_i, $i = 1, 2$ or 3, the aforementioned authors constructed families of functions $\mathscr{F}(m, n, \varepsilon) \subseteq C^r[-1, 1] \backslash \pi^r_{m,n}$, where $(m, n) \in R_i$ and where $\varepsilon > 0$, with the property that

$$\gamma_{m,n} = \inf\{E^c_{m,n}(f)/E^r_{m,n}(f) : f \in \mathscr{F}(m, n, \varepsilon) \text{ and } \varepsilon > 0\}.$$

In this section, we give a unified approach to calculating a sharp upper bound for $\gamma_{m,n}$ in each of the regions R_1, R_2, and R_3 of Table 1. In addition to providing a consistent framework for calculating upper bounds of $\gamma_{m,n}$, the details presented below also provide the foundation required for the sharpness results given in Theorem 4.

Our first result provides a new tool for obtaining upper bounds for $\gamma_{m,n}$.

Proposition 1. *For a fixed pair* (m, n) *of nonnegative integers, let*

$$\phi \in (\pi^c_{m,n} \backslash \pi^r_{m,n}) \cap C^r[-1, 1],$$

and let S *be a continuous real-valued function on* $[-1, 1]$ *for which there are* $L \geq m+2$ *distinct points* $\{x_j\}_{j=1}^L$, *with* $-1 \leq x_1 < x_2 < \cdots < x_L \leq 1$, *such that*

$$(2.1) \qquad (-1)^j \delta(S(x_j) + \operatorname{Re} \phi(x_j)) > 0 \qquad (j = 1, 2, \ldots, L),$$

where δ *is a constant which is either* $+1$ *or* -1. *Then,*

$$(2.2) \qquad \gamma_{m,n} \leq \|S - i \operatorname{Im} \phi\|_I / M,$$

where

$$M := \min_{1\le j\le L} |S(x_j) + \operatorname{Re}\phi(x_j)|. \tag{2.3}$$

Proof. Set $f(x) := S(x) + \operatorname{Re}\,\phi(x)$. Then, as condition (2.1) states that the error function for the zero approximation to f oscillates in $L \ge m+2$ points, the de la Vallée Poussin Theorem [2, p. 83] gives (cf. (2.3)) that $E^r_{m,n}(f) \ge M$. But as $E^c_{m,n}(f) \le \|f - \phi\|_I = \|S - i\,\operatorname{Im}\,\phi\|_I$, we must have from (1.3) that $\gamma_{m,n} \le \|S - i\,\operatorname{Im}\,\phi\|_I/M$. □

Given a pair of nonnegative integers (m, n) with $n \ge 1$, Proposition 1 suggests a procedure for finding a sequence of functions $\{f_\varepsilon\} \subseteq C^r[-1, 1]\backslash\pi^r_{m,n}$ for which $E^c_{m,n}(f_\varepsilon)/E^r_{m,n}(f_\varepsilon)$ is minimized. One first chooses a continuous rational function $\phi_\varepsilon \in \pi^c_{m,n}\backslash\pi^r_{m,n}$ on $[-1, 1]$ with the property that $\operatorname{Re}\,\phi_\varepsilon(x)$ has at least $m+1$ sign changes in $[-1, 1]$ and for which $\|\operatorname{Im}\,\phi_\varepsilon\|_I$ is small. Such a function may be obtained (see Theorems 2, 3, and 4 below) by placing, in an astute manner, the zeros and poles of ϕ_ε near the interval $[-1, 1]$. Suppose $(-1)^j\operatorname{Re}\,\phi_\varepsilon(x_j) > 0$ $(j = 1, 2, \dots, m+2)$, where $-1 \le x_1 < \cdots < x_{m+2} \le 1$. The function S_ε is then chosen so that

$$\operatorname{sgn} S_\varepsilon(x_j) = \operatorname{sgn} \operatorname{Re}\ \phi_\varepsilon(x_j) \quad (j = 1, 2, \dots, m+2), \tag{2.4}$$

$$|S_\varepsilon(x_j) - i\,\operatorname{Im}\,\phi_\varepsilon(x_j)| \approx \|\operatorname{Im}\,\phi_\varepsilon\|_I \qquad (j = 1, 2, \cdots, m+2), \tag{2.5}$$

and

$$S_\varepsilon(x) = 0, \text{ for } x \notin \bigcup_{j=1}^{m+2} (x_j - \varepsilon,\ x_j + \varepsilon), \text{ for some sufficiently small } \varepsilon > 0. \tag{2.6}$$

The condition of (2.4) is used to make

$$M := \min_{1\le j\le m+2} \{|S_\varepsilon(x_j) + \operatorname{Re}\ \phi(x_j)|\}$$

as large as possible, while conditions (2.5) and (2.6) are used to guarantee that $\|S_\varepsilon - i\,\operatorname{Im}\,\phi_\varepsilon\|_I \approx \|\operatorname{Im}\,\phi_\varepsilon\|_I$. These choices make the ratio $\|S_\varepsilon - i\,\operatorname{Im}\,\phi_\varepsilon\|_I/M$ in (2.2) nearly as small as possible.

As a concrete example of the above procedure, consider the integer pair $(0, 2)$ and, for any $\varepsilon > 0$ sufficiently small, set

$$\phi_\varepsilon(x) := \frac{2\varepsilon i}{3}\left[\frac{1}{x+1-i\varepsilon} - \frac{1}{x-1-i\varepsilon}\right],$$

$$h(x) := \begin{cases} \dfrac{1-x^2}{1+x^2}, & x \in [-1, 1], \\ 0 & \text{otherwise}, \end{cases}$$

and

$$S_\varepsilon(x) := \frac{1}{3}\left[h\left(\frac{x-1}{\varepsilon}\right) - h\left(\frac{x+1}{\varepsilon}\right)\right].$$

The function $\phi_\varepsilon(x)$, an element of $\pi^c_{0,2}\backslash\pi^r_{0,2}$, can be verified to satisfy

$$\text{Re }\phi_\varepsilon(-1) = -\tfrac{2}{3} + O(\varepsilon^2) < 0, \quad \text{and} \quad \text{Re }\phi_\varepsilon(1) = \tfrac{2}{3} + O(\varepsilon^2) > 0,$$

for all $\varepsilon > 0$ sufficiently small. Next, on setting $x_1 := -1$ and $x_2 := +1$, the function $S_\varepsilon(x)$ then directly satisfies (2.4) and (2.6), and, as a short calculation shows, it also satisfies (2.5), up to an additive term $O(\varepsilon)$, i.e.,

$$\tfrac{1}{3} + O(\varepsilon^2) = \left|S_\varepsilon(x_j) - i \text{ Im } \phi_\varepsilon(x_j)\right| = \|\text{Im } \phi_\varepsilon\|_I + O(\varepsilon) \qquad (j = 1, 2).$$

In addition, it can be similarly verified that

$$\|S_\varepsilon - i \text{ Im } \phi_\varepsilon\|_I = \tfrac{1}{3} + O(\varepsilon),$$

and

$$M := \min_{j=1,2}\left\{\left|S_\varepsilon(x_j) + \text{Re } \phi_\varepsilon(x_j)\right|\right\} = 1 + O(\varepsilon^2).$$

By (2.2) of Proposition 1, we thus have the upper bound

$$\gamma_{0,2} \le \frac{\|S_\varepsilon - i \text{ Im } \phi_\varepsilon\|_I}{M} = \frac{1}{3} + O(\varepsilon),$$

for all $\varepsilon > 0$ sufficiently small, whence on letting $\varepsilon \to 0$,

$$\gamma_{0,2} \le \tfrac{1}{3}.$$

To establish the known upper bounds for $\gamma_{m,n}$ associated with the regions R_i, $i = 1, 2$, and 3, of Table 1, the authors of [1], [3], and [5] each, in essence, applied a variant of Proposition 1, with appropriate choices for ϕ_ε and S_ε, to obtain upper bounds for $\gamma_{m,n}$. Normalized forms of their choices of ϕ_ε and S_ε are detailed in the next three theorems. For notation, $\prod_{j=1}^m d_j := 1$ when $m \le 0$.

Theorem 1 (Trefethen and Gutknecht [5]). *For any $m \ge 0$ and $\varepsilon > 0$ sufficiently small, set*

$$g_{m,\varepsilon}(x) := \frac{\varepsilon \prod_{j=1}^m [-1 + (2j-1)\varepsilon - x]}{[x + (1+\varepsilon)]^{m+1}\left(i\sqrt{\varepsilon} - x\right)(1 + \varepsilon - x)}, \tag{2.7}$$

so that $g_{m,\varepsilon} \in \pi^c_{m,m+3}\backslash\pi^r_{m,m+3}$, and set

$$\phi_{m,\varepsilon}(x) := g_{m,\varepsilon}(x)/\|\text{Im } g_{m,\varepsilon}(x)\|_I, \quad \textit{and} \quad S_\varepsilon(x) := 0.$$

Then, there is a constant $c > 0$, independent of ε, such that for all $\varepsilon > 0$ sufficiently small, there are $m+2$ distinct points $\{x_j(\varepsilon)\}_{j=1}^{m+2}$, with $-1 \le x_1(\varepsilon) < x_2(\varepsilon) < \cdots < x_{m+2}(\varepsilon) \le 1$, for which

$$(-1)^j \text{Re } \phi_{m,\varepsilon}(x_j(\varepsilon)) \ge c/\sqrt{\varepsilon} \qquad (j = 1, 2, \ldots, m+2), \tag{2.8}$$

$$\|S_\varepsilon - i \text{ Im } \phi_{m,\varepsilon}\|_I = \|\text{Im } \phi_{m,\varepsilon}\|_I = 1, \tag{2.9}$$

and

$$M := \min_{1\le j\le m+2}|S_\varepsilon(x_j(\varepsilon)) + \text{Re } \phi_{m,\varepsilon}(x_j(\varepsilon))| \ge c/\sqrt{\varepsilon}. \tag{2.10}$$

Theorem 2 (Levin [1]). *For any nonnegative integers* n *and* k *with* $n \geq 2$ *and* k *even, set*

$$g_{k,n,\varepsilon}(x) := T_k(x) \cdot \left(\frac{x - i\varepsilon}{x + i\varepsilon}\right)^n \tag{2.11}$$

where $T_k(x)$ *is the normalized (i.e.,* $\|T_k\|_I = 1$*) Chebyshev polynomial of the first kind of degree* k*, and set* $\phi_{k,n,\varepsilon}(x) := g_{k,n,\varepsilon}(x)/\|\text{Im } g_{k,n,\varepsilon}(x)\|_I$ *and* $S_\varepsilon(x) := S_{k,n,\varepsilon}(x) := \text{Re } \phi_{k,n,\varepsilon}(x)$*. Then, there is a constant* $c > 0$*, independent of* ε*, such that for all* $\varepsilon > 0$ *sufficiently small, there are* $k + 2n + 1$ *distinct points,* $\{x_j(\varepsilon)\}_{j=1}^{k+2n+1}$*, with* $-1 \leq x_1(\varepsilon) < x_2(\varepsilon) < \cdots < x_{k+2n+1}(\varepsilon) \leq 1$*, for which*

$$1 - c\varepsilon \leq (-1)^{j+1} \text{Re } \phi_{k,n,\varepsilon}(x_j(\varepsilon)) \leq \|S_{k,n,\varepsilon} - i \text{ Im } \phi_{k,n,\varepsilon}\|_I \leq 1 + c\varepsilon, \tag{2.12}$$

and

$$M := \min_{1 \leq j \leq k+2n+1} |S_{k,n,\varepsilon}(x_j(\varepsilon)) + \text{Re } \phi_{k,n,\varepsilon}(x_j(\varepsilon))| \geq 2 - 2c\varepsilon. \tag{2.13}$$

Theorem 3 (Ruttan and Varga [3]). *For any* $m \geq 0$*, let*

$$g_{m,\varepsilon}(x) := \frac{-2\varepsilon i}{3} \sum_{j=0}^{m+1} \frac{\mu_j(-1)^j}{x - 1 + \frac{2j}{m+1} - \varepsilon\mu_j i} \tag{2.14}$$

where $\{\mu_j\}_{j=0}^{m+1}$ *are any* $m + 2$ *fixed positive numbers satisfying*

$$0 < \mu_j \leq 1, \qquad \sum_{j=0}^{m+1} (-1)^j \mu_j = 0 \quad \text{and} \quad \sum_{j=0}^{m+1} j(-1)^j \mu_j \neq 0,$$

so that $g_{m,\varepsilon} \in \pi^c_{m,m+2} \backslash \pi^r_{m,n+2}$*, and let*

$$h(x) = \begin{cases} \dfrac{1 - x^2}{1 + x^2}, & x \in [-1, 1], \\ 0, & \textit{otherwise.} \end{cases}$$

Set $\phi_{m,\varepsilon}(x) := g_{m,\varepsilon}(x)/\|\text{Im} g_{m,\varepsilon}(x)\|_I$ *and*

$$S_\varepsilon(x) := S_{m,\varepsilon}(x) := \left(\sum_{j=0}^{m+1} (-1)^j h\left(\frac{x - 1 + \frac{2j}{m+1}}{\varepsilon}\right)\right) / \|\text{Im } g_{m,\varepsilon}(x)\|_I.$$

Then, there is a constant $c > 0$*, independent of* ε *such that for all* $\varepsilon > 0$ *sufficiently small, there are* $m+2$ *distinct points* $\{x_j(\varepsilon)\}_{j=1}^{m+2}$*, with* $-1 \leq x_1(\varepsilon) < x_2(\varepsilon) < \cdots < x_{m+2}(\varepsilon) \leq 1$*, for which*

$$(-1)^j \delta \text{ Re } \phi_{m,\varepsilon}(x_j(\varepsilon)) \geq 2 - c\varepsilon \qquad (j = 1, 2, \ldots, m + 2), \tag{2.15}$$

where δ *is a constant which is either* $+1$ *or* -1*,*

$$\|S_{m,\varepsilon} - i \text{ Im } \phi_{m,\varepsilon}\| < 1 + c\varepsilon, \tag{2.16}$$

and

(2.17) $$M := \min_{1\le j\le m+2} \left|S_{m,\varepsilon}(x_j(\varepsilon)) + \operatorname{Re}\phi_{m,\varepsilon}(x_j(\varepsilon))\right| > 3 - c\varepsilon.$$

On combining the results of (2.9) and (2.10) of Theorem 1 with (2.2) of Proposition 1, it is evident that $0 \le \gamma_{m,m+3} \le \sqrt{\varepsilon}/c$ for all $\varepsilon > 0$ sufficiently small, so that (cf. Trefethen and Gutknecht [5])

$$\gamma_{m,m+3} = 0 \qquad (m \ge 0).$$

But as $\pi^c_{m,m+k} \supseteq \pi^c_{m,m+3}$ for all $k \ge 3$, the same function $\phi_{m,\varepsilon}$ of Theorem 1 can be used to deduce (as was pointed out in [5]) that

$$\gamma_{m,n} = 0 \quad (\text{all } n \ge m+3; m \ge 0).$$

In a similar fashion, on combining the results of Theorems 2 and 3 with Proposition 1 gives the upper bounds of

(2.18) $$\gamma_{m,n} \le \tfrac{1}{2} \quad (m+1 \ge n \ge 1); \qquad \gamma_{m,m+2} \le \tfrac{1}{3} \quad (m \ge 0).$$

(We remark that the case $n = 1$ of the first inequality of (2.18) requires special handling. For details, see Levin [1].)

3. Oscillation of the real part of a rational function

For a given real or complex polynomial p, let ∂p denote the exact degree of p. If $R = p/q$ is continuous on $[-1, 1]$ where p and q are real polynomials, it is evident that $\operatorname{Re} R = R$ can have at most ∂p sign changes (as, for example in (2.1)) since each sign change of R corresponds to a zero of p. But, what can be said about the number of sign changes when $R = p/q$ is a continuous *complex-valued* rational function on $[-1, 1]$? As we shall show in our next theorem, the number of possible sign changes of $\operatorname{Re} R$ depends not only on the degrees of p and q, but also on the size of the oscillations of $\operatorname{Re} R$. For additional notation, let $\lfloor x \rfloor$ denote the greatest integer N satisfying $N \le x$. Then, we have the new result of

Theorem 4. *Let $\phi = p/q$ be a complex rational function with no poles in $[-1, 1]$ which satisfies $\|\operatorname{Im}\phi\|_I \le 1$. Assume that there are real numbers $d > 0$ and $\{x_j\}_{j=1}^L$, with $-1 \le x_1 < x_2 < \cdots < x_L \le 1$, for which*

(3.1) $$\delta(-1)^j \operatorname{Re}\phi(x_j) \ge d \qquad (j = 1, 2, 3, \ldots, L),$$

where δ is a constant which is either $= +1$ or -1. If $\partial q \le \partial p$ and if $d \ge 1$, then

(3.2) $$L \le \partial p + 1.$$

Similarly, if $\partial q > \partial p$, then

(3.3) $$L \le \partial q \quad \text{whenever } d \ge 1,$$

and

$$L \leq \left\lfloor \frac{\partial p + \partial q + 1}{2} \right\rfloor \quad \text{whenever } d \geq 2. \tag{3.4}$$

The upper bounds for L *given in* (3.2)–(3.4) *are sharp in the following senses:*

(3.5) $\left\{\begin{array}{l}\textit{there exist rational functions, satisfying the ap-}\\ \textit{propriate hypotheses, for which the upper bonds}\\ \textit{for } L \textit{ given in } (3.2)\text{–}(3.4) \textit{ are attained } (\textit{i.e., equal-}\\ \textit{ity can hold in } (3.2)\text{–}(3.4));\end{array}\right.$

(3.6) $\left\{\begin{array}{l}\textit{for any } d<1 \ (d<2 \textit{ respectively}), \textit{ there exists}\\ \textit{rational functions satisfying all but the hypothe-}\\ \textit{ses on } d \textit{ in } ,(3.3)\ ((3.4) \textit{ respectively}) \textit{ for which the}\\ \textit{bound on } L \textit{ is exceeded.}\end{array}\right.$

Proof. As the proofs associated with (3.5) and (3.6) are more direct, we first consider the sharpness results expressed in (3.5) and (3.6). To establish (3.5), we must exhibit rational function ϕ_1, ϕ_2, and ϕ_3 which satisfy the hypotheses for (3.2), (3.3) and (3.4), respectively. To that end, first set $\phi_1 = p/q$ where $q(x) := 1$, and set $p(x) := T_m(x) = \cos(m \arccos x)(\text{for } -1 \leq x \leq 1)$. Then, $\partial p = m$, $0 = \|\text{Im } \phi_1\|_I \leq 1$, and, with the known $m+1$ extremal points $\{\hat{x}_j := \cos(j\pi/m)\}_{j=0}^m$ for the Chebyshev polynomial $T_m(x)$, (i.e., $T_m(\hat{x}_j) = (-1)^j$), then (3.1) is valid for $d = 1$ and for the $L := m+1$ points $\{\hat{x}_j\}_{j=0}^m$. In this case, equality then holds in (3.2). To verify that equality is attainable in (3.3), let $\phi_2 = \phi_{m,\varepsilon} = p/q$ where $\phi_{m,\varepsilon}$ is given in Theorem 3. By Theorem 3, ϕ_2 (an element of $\pi^c_{m,m+2} \backslash \pi^r_{m,m+2}$) satisfies (3.1) with $d \geq 1$ whenever ε is sufficiently small. But $\partial p = m$, $\partial q = m+2$, and $L = m+2$, so, consequently, equality also can hold in (3.3).

It remains to verify that equality can hold in (3.4). Let $\phi = p/q$ be the rational function $\phi_{m,\varepsilon}$ given in Theorem 1. By (2.8), ϕ_3 (an element of $\pi^c_{m,m+3} \backslash \pi^r_{m,m+3}$) satisfies (3.1) with $L = m+2$ and $d \geq 2$, provided $\varepsilon > 0$ is sufficiently small. Since $\partial p = m$ and $\partial q = m+3$, we have $L = m+2 = \lfloor \frac{\partial p + \partial q + 1}{2} \rfloor$, which shows that equality can hold in (3.4). This completes the proof of the sharpness in (3.5)

To establish the claimed sharpness (cf. (3.6)) of (3.4), consider first the function $\phi_{m,\varepsilon}$ (in $\pi^c_{m,m+2} \backslash \pi^r_{m,m+2}$) of Theorem 3. From Theorem 3, we see that $\phi_{m,\varepsilon} = p/q$ satisfies $\|\text{Im } \phi_{m,\varepsilon}\|_I \leq 1$ and hypothesis (3.1) of Theorem 4 with $L = m+2$ and $d < 2$ (for all $\varepsilon > 0$ sufficiently small). But in this case, as $\partial p = m$, and as $\partial q = m+2$, then $L = m+2 > \lfloor \frac{\partial p + \partial q + 1}{2} \rfloor$, which shows that the inequality of (3.4) of Theorem 4 can fail if the condition $d \geq 2$ is deleted. In a similar constructive manner, using $\phi_\varepsilon(x) = \phi_{k,n,\varepsilon}(x)/(1+\varepsilon x)^{k+1}$ where $\phi_{k,n,\varepsilon}$ is defined in Theorem 2, one obtains the sharpness, as claimed in (3.6), for the inequality of (3.3).

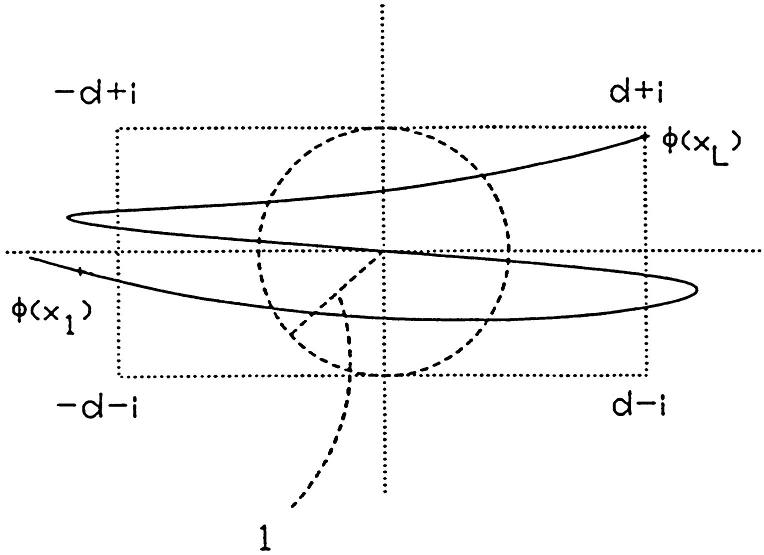

FIGURE 1

We now establish (3.2)–(3.4). We remark that inequalities (3.2) and (3.3) can be deduced from the results found in [1], but for completeness we include a proof here.

To establish (3.2) of Theorem 4, we use a geometrical argument, suggested by the work of Levin [1]. Assume $d \geq 1$, and consider a circle $C := \{z : |z| = 1\}$ and a rectangle B with vertices $\pm d \pm i$ as indicated in Figure 1. Condition (3.1) and the assumption that $\|\text{Im}\ \phi\|_I \leq 1$ imply that the curve (in the extended plane) $\Gamma_1 := \{z = \phi(x) : x \in (-\infty, \infty)\}$ intersects the vertical sides of B, and, hence the circle C in $2(L-1)$ points as x increases from x_1 to x_L. (Here, points where Γ_1 is tangent to C are counted twice.) If x gives such an intersection of the curve Γ_1 and C, i.e.,

$$|\phi(x)|^2 = \left|\frac{p(x)}{q(x)}\right|^2 = 1,$$

then x is also a zero of the polynomial

$$P(x) := |p(x)|^2 - |q(x)|^2. \tag{3.7}$$

The above discussion shows that there are at least $2(L-1)$ zeros of $P(x)$ in $[x_1, x_L]$.

If $\partial p \geq \partial q$, then $P(x)$ of (3.7) is a polynomial in x with degree at most $2\partial p$. Therefore, it must follow that $2(L-1) \leq \partial P(x) \leq 2\partial p$, from which we obtain (3.2).

Next, to establish (3.3) of Theorem 4, assume that hypothesis (3.1) is valid, that $\partial q > \partial p$, and that $d \geq 1$. As in the previous case, we know that that

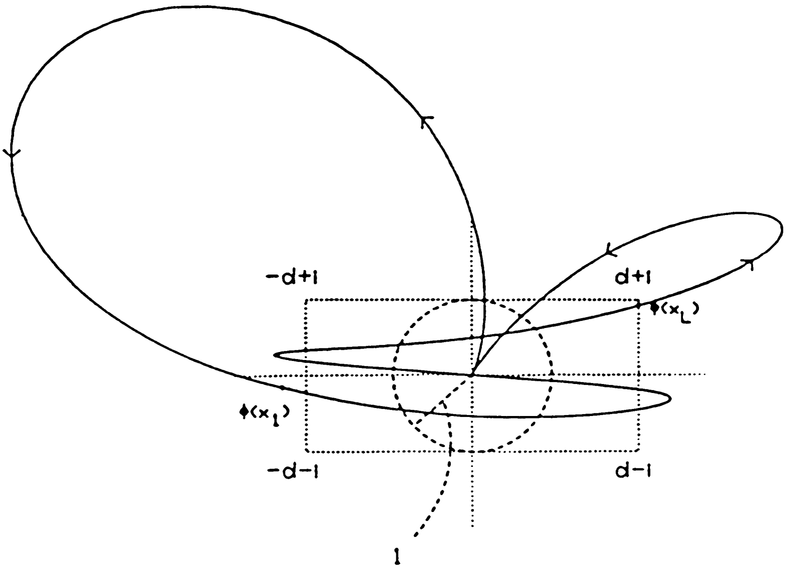

FIGURE 2

portion of the curve Γ_1, as x increases from x_1 to x_L, intersects the circle C at least $2(L-1)$ times. Since $d \geq 1$, it is geometrically clear that $\phi(x_1)$ and $\phi(x_L)$ both lie *outside* of C (cf. (3.1)) if any of the following statements is valid:

$$(3.8) \qquad \begin{cases} \text{(i)} & d > 1; \\ \text{(ii)} & \delta(-1)\text{Re}\, \phi(x_1) > 1 \text{ and } \delta(-1)^L \text{Re}\, \phi(x_L) > 1; \\ \text{(iii)} & \text{Im}\, \phi(x_1) \neq 0 \neq \text{Im}\, \phi(x_L). \end{cases}$$

But, in this case (i.e., $\partial q > \partial p$), it follows that $\phi(x) \to 0$ as $x \to \pm\infty$. As 0 is an *interior* point of C, then there is evidently an additional intersection of Γ_1 and C in each of the intervals $(-\infty, x_1)$ and $(x_L, +\infty)$. (This is illustrated in Figure 2.) Thus, $P(x)$ of (3.7) must have a total of at least $2L$ zeros. As $\partial q > \partial p$, then $\partial P = 2\partial q$, so that $2L \leq 2\partial q$. This establishes (3.3) whenever $\phi(x_1)$ and $\phi(x_2)$ both lie outside of C.

For the remaining case, suppose (in contrast with equations (3.8)) that $\delta(-1)\phi(x_1) = 1 = d$ and, for convenience, assume $\delta = +1$, so that $\phi(x_1) = -1$. If Γ_1 is *not* tangent to C at -1 (this possibility is shown on the left of Figure 3), then it is possible to find a real $\tilde{x}_1$ sufficiently near x_1 for which $-\text{Re}\, \phi(\tilde{x}_1) > 1$ and $\|\text{Im}\, \phi\|_{[\tilde{x}_1, +1]} \leq 1$ are both satisfied. With a possible linear change in scale (mapping $[\tilde{x}_1, +1]$ into $[-1, +1]$), then $\phi(\tilde{x}_1)$ is outside C, and the previous argument can be applied. Finally, if Γ_1 *is* tangent to C at $x = 1$ (as indicated on the right of Figure 3), this contact implies that $x = 1$ is a zero of multiplicity at least two of $P(x)$, and we conclude in all cases that $P(x)$ must have at least $2L$ zeros, which gives (3.3).

Now, for the remaining inequality (3.4) of Theorem 4, assume $\partial q > \partial p$ and $d \geq 2$. Again, consider the rectangle B with vertices $\pm d \pm i$. The assumption

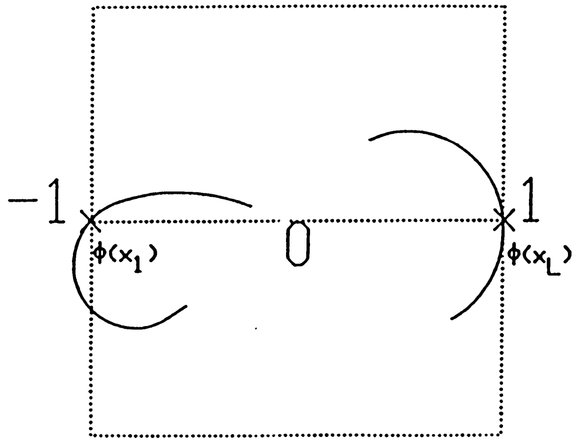

FIGURE 3

that $d \geq 2$ means that the circles $C_1 := \{z : |z+1| = 1\}$ and $C_2 := \{z : |z-1| = 1\}$ each lie within the rectangle B. As in the cases above, condition (3.1) and the assumption that $\|\text{Im } \phi\|_I \leq 1$, imply that that portion of the curve Γ_1 intersects C_1 in $2(L-1)$ points as x increases from x_1 to x_L. (Again, points of tangency are counted twice). Let $\{v_j\}_{j=1}^{2(L-1)}$ be the $2(L-1)$ points with $x_1 \leq v_1 \leq v_2 \leq \cdots \leq v_{2(L-1)} \leq x_L$ for which $\{\phi(v_j)\}_{j=1}^{2(L-1)}$ lie on C_1. Thus, the points $v_1, v_2, \ldots, v_{2(L-1)}$ satisfy

$$|\phi(v_j) + 1|^2 = 1 \qquad (j = 1, 2, \ldots, 2(L-1)). \tag{3.9}$$

Similarly, we see that Γ_1 intersects C_2 in $2(L-1)$ points. Let $\{u_j\}_{j=1}^{2(L-1)}$ be the $2(L-1)$ points with $x_1 \leq u_1 \leq u_2 \leq \cdots \leq u_{2(L-1)} \leq x_L$ for which $\{\phi(u_j)\}_{j=1}^{2(L-1)}$ lies on C_2. This situation is illustrated in Figure 4.

Currently, the polynomials p and q are determined only up to a multiplicative constant. So, without loss of generality, we may assume that

$$p(x) = \prod_{j=1}^{\partial p}(x - \alpha_j) \quad \text{and} \quad q(x) = \beta\prod_{j=1}^{\partial q}(x - \beta_j) \qquad (\beta \neq 0), \tag{3.10}$$

where $\{\alpha_j\}_{j=1}^{\partial p}$ are the zeros of ϕ and $\{\beta_j\}_{j=1}^{\partial q}$ are the poles of ϕ. With this representation, (3.9) implies that

$$P_1(x) := |p(x) + q(x)|^2 - |q(x)|^2 = 2 \text{ Re } \left\{\overline{p(x)}q(x)\right\} + |p(x)|^2 \tag{3.11}$$

has $2(L-1)$ zeros $\{v_j\}_{j=1}^{2(L-1)}$ in $[x_1, x_L]$. And similarly,

$$P_2(x) := |p(x) - q(x)|^2 - |q(x)|^2 = -2 \text{ Re } \left\{\overline{p(x)}q(x)\right\} + |p(x)|^2 \tag{3.12}$$

has $2(L-1)$ zeros $\{u_j\}_{j=1}^{2(L-1)}$ in $[x_1, x_L]$. How the proof now proceeds depends on the sign of Re β. If Re $\beta < 0$ we will find that $P_1(x)$ has enough

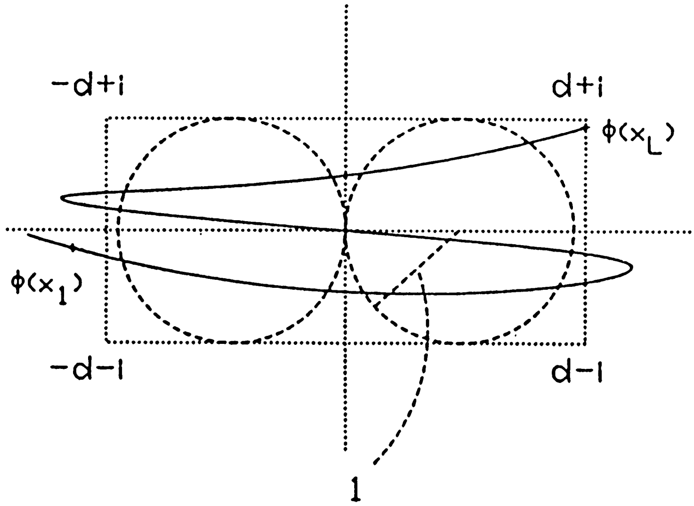

FIGURE 4

additional zeros to establish (3.4). If $\operatorname{Re}\beta \geq 0$, $P_2(x)$ can be used to establish (3.4). We treat only the case $\operatorname{Re}\beta \geq 0$, the case $\operatorname{Re}\beta < 0$ being completely similar. Our goal is to find two additional zeros for $P_2(x)$ when $K := \partial q - \partial p$ is an even positive integer and one additional zero when K is an odd positive integer. Note, first, that if Γ_1 is tangent to C_2 at x_1, then x_1 is a zero of $P_2(x)$ with multiplicity at least 2. In that case, we have an additional zero associated with x_1. In a similar fashion, we find an additional zero associate with x_L if Γ_1 is tangent to C_2 at x_L.

There are three cases to consider: K even and $\operatorname{Re}\beta > 0$, K odd and $\operatorname{Re}\beta > 0$, and $\operatorname{Re}\beta = 0$.

Case 1: K even and $\operatorname{Re}\beta > 0$. As we observed above, if Γ_1 is tangent to C_2 at x_1, then there is an additional zero of $P_2(x)$ associated with x_1. If Γ_1 is not tangent to C_2 at x_1, then since $|\operatorname{Re}\phi(x_1)| \geq d \geq 2$ we proceed as in the proof of (3.3) to show that there is a real $\tilde{x}_1$ arbitrarily near x_1 (and possibly equal to x_1) for which $\operatorname{sgnRe}\phi(x_1) = \operatorname{sgnRe}\phi(\tilde{x}_1)$ and $|\operatorname{Re}\phi(\tilde{x}_1)| > d$. If one replaces x_1 with $\tilde{x}_1$, then the hypotheses of the theorem still hold (after a possible linear change in scale). Therefore, without loss of generality, we may assume $|\operatorname{Re}\phi(x_1)| > d \geq 2$, and hence $|\phi(x_1)| = \left|\frac{p(x_1)}{q(x_1)}\right| > d \geq 2$. Consequently, it follows that

$$\left|\operatorname{Re}\frac{q(x_1)}{p(x_1)}\right| \leq \left|\frac{q(x_1)}{p(x_1)}\right| < \frac{1}{2}. \tag{3.13}$$

Using the (3.10), we see that

$$\operatorname{Re}\frac{q(x)}{p(x)} = (\operatorname{Re}\beta)x^K + \text{ lower order terms in } x. \tag{3.14}$$

Since $\operatorname{Re}\beta > 0$ and K is an even positive integer, then as $x \to -\infty$, (3.14) shows that $\operatorname{Re}\frac{q(x)}{p(x)} \to +\infty$. This together with (3.13) establishes that there is an $\hat{x}$ in $(-\infty, x_1)$ for which

$$\operatorname{Re}\frac{q(\hat{x})}{p(\hat{x})} = \frac{1}{2}. \tag{3.15}$$

But (3.15) may be rewritten as

$$-2\operatorname{Re}\left\{\overline{p(\hat{x})}q(\hat{x})\right\} + |p(\hat{x})|^2 = 0,$$

which shows that $P_2(x)$ has a zero in $(-\infty, x_1)$, when Γ_1 is not tangent to C_2. So, in either case (Γ_1 tangent to C_2, or Γ_1 not tangent to C_2), we find an extra zero associated with x_1. Similarly, we find an extra zero associated with x_L. Thus, when K is an even positive integer and $\operatorname{Re}\beta > 0$, $P_2(x)$ has $2L$ zeros. But then,

$$2L \le \partial P_2 \le \partial p + \partial q \le \partial p + \partial q + 1.$$

Hence,

$$L \le \left\lfloor \frac{\partial p + \partial q + 1}{2} \right\rfloor,$$

which establishes (3.4) for this case.

Case 2: K odd and $\operatorname{Re}\beta > 0$. If Γ_1 is tangent to C_2 at x_L, the tangency then gives the required additional zero. When Γ_1 is not tangent to C_2 at x_L then, after a possible substitution of x_L with a point $\tilde{x}_L$ sufficiently close to x_L, followed by a possible linear substitution, we find that $|\phi(x_L)| > 2$, from which

$$\left|\operatorname{Re}\frac{q(x_L)}{p(x_L)}\right| < \frac{1}{2} \tag{3.16}$$

follows. As $\operatorname{Re}\beta > 0$, (3.14) shows that $\operatorname{Re}\frac{q(x)}{p(x)} \to +\infty$ as $x \to +\infty$. Arguing as in Case 1, this together with (3.16) yields that $P_2(x)$ has an additional zero in $(x_L, +\infty)$. Thus, we find that $P_2(x)$ has $2L - 1$ zeros, and therefore

$$2L \le \partial P_2 + 1 \le \partial p + \partial q + 1.$$

Consequently,

$$L \le \left\lfloor \frac{\partial p + \partial p + 1}{2} \right\rfloor.$$

Case 3: $\operatorname{Re}\beta = 0$. Since $\partial q > \partial p$, it follows from (3.10) and (3.12) that

$$\begin{aligned} P_2(x) &= -2\operatorname{Re}(\overline{p(x)}q(x)) + |p(x)|^2 \\ &= -2(\operatorname{Re}\beta)x^{\partial q + \partial p} + (\operatorname{Re}\mu)x^{\partial q + \partial p - 1} + \text{lower order terms}, \end{aligned}$$

for some constant μ. But, as $\operatorname{Re}\beta = 0$, we have that $2L - 2 \le \partial P_2 \le \partial p + \partial q - 1$. Therefore

$$L \le \left\lfloor \frac{\partial p + \partial q + 1}{2} \right\rfloor,$$

which gives (3.4). □

4. Lower bounds for $\gamma_{m,n}$

With the aid of Theorem 4, we will now establish that $\gamma_{m,m+2} = 1/3$ for all $m \geq 0$, and show that the previously mentioned lower bounds for $\gamma_{m,n}$, $m \geq n+2$, hold.

Theorem 5. *Let (m, n) be a pair on nonnegative integers with $n \geq 1$, let $f \in C^r[-1, 1]\backslash\pi^r_{m,n}$, and let r and R be respectively the best uniform approximation of f on $[-1, 1]$ from $\pi^r_{m,n}$ and $\pi^c_{m,n}$. Then,*

$$\|f - R\|_I/\|f - r\|_I > 1/2 \quad \text{if } m+1 \geq n \tag{4.1}$$

and

$$\|f - R\|_I/\|f - r\|_I > 1/3 \quad \text{if } m+2 \geq n. \tag{4.2}$$

Hence,

$$\gamma_{m,n} = 1/2 \qquad \text{if } m+1 \geq n, \tag{4.3}$$

and

$$\gamma_{m,n} = 1/3 \qquad \text{if } m+2 = n. \tag{4.4}$$

Proof. Let $S := \|f-R\|_I/\|f-r\|_I$. Set $e := f-r$, $R := p_1/q_1$, and $r := p_2/q_2$ where the pairs (p_1, q_1) and (p_2, q_2) are assumed to have no common factors. Since $f \notin \pi^r_{m,n}$, then by multiplying f, r, and R by an appropriate constant, we may assume that $\|e\|_I = 1$. As r is the best uniform approximant of f, there exist at least $L := m+n+2-\min(m-\partial p_2; n-\partial q_2)$ distinct points $\{x_j\}_{j=1}^L$, with $-1 \leq x_1 < x_2 < \cdots < x_L \leq 1$, such that $e(x_j) = (-1)^j\delta$ for all $1 \leq j \leq L$, where δ is a constant which is either $+1$ or -1. Again, on multiplying by -1, if necessary, we may take $\delta = 1$, i.e., $e(x_1) = -1$.

With this normalization,

$$S := \|f - R\|_I \geq |f(x_j) - R(x_j)| = |(-1)^j + r(x_j) - R(x_j)| \qquad (1 \leq j \leq L),$$

which is possible only if

$$(-1)^j \text{Re}\,(R(x_j) - r(x_j)) \geq 1 - S \qquad (1 \leq j \leq L). \tag{4.5}$$

Let $\phi(x) := (R(x) - r(x))/S := p(x)/q(x)$ where p and q are polynomials with no common factors. Then, as

$$S = \|f - R\|_I = \|e - R + r\|_I \geq \|\text{Im}\,(e - R + r)\|_I = \|\text{Im}\,R\|_I, \tag{4.6}$$

(4.5) implies

$$(-1)^j \text{Re}\,\phi(x_j) \geq \frac{1-S}{S} =: d \qquad (1 \leq j \leq L), \tag{4.7}$$

and (4.6) implies

$$\|\text{Im}\,\phi\|_I \leq 1. \tag{4.8}$$

To establish (4.1) of Theorem 5, it suffices to establish the contrapositive of (4.1) i.e., if $S \le \frac{1}{2}$, then $m+1 < n$, or equivalently

$$\text{if } S \le \tfrac{1}{2}, \text{ then } m+2 \le n. \tag{4.9}$$

Similarly, to establish (4.2) of Theorem 5, it suffices to establish that if $S \le \frac{1}{3}$, then $m+2 < n$, or equivalently

$$\text{if } S \le \tfrac{1}{3}, \text{ then } m+3 \le n. \tag{4.10}$$

To this end, first assume that $S \le 1/2$. Then from (4.7), $d \ge 1$, and on applying Theorem 4 to (4.7) and (4.8), we obtain

$$L \le \partial p + 1 \quad \text{if } \partial p \ge \partial q, \tag{4.11}$$

and

$$L \le \partial q \quad \text{if } \partial p < \partial q. \tag{4.12}$$

Since

$$\begin{aligned}\phi(x) = p(x)/q(x) &= (R(x) - r(x))/S \\ &= \frac{p_1(x)q_2(x) - p_2(x)q_1(x)}{Sq_1(x)q_2(x)},\end{aligned}$$

it follows that

$$\begin{cases} \partial p \le \max(\partial p_1 + \partial q_2 ; \partial p_2 + \partial q_1), \text{ and} \\ \partial q = \partial q_1 + \partial q_2. \end{cases} \tag{4.13}$$

If $\partial p \ge \partial q$, then (4.11) holds and thus

$$\begin{aligned} &m + n + 2 - \min(m - \partial p_2 ; n - \partial q_2) \\ &\qquad =: L \le \partial p + 1 \le \max(\partial p_1 + \partial q_2, \partial p_2 + \partial q_1) + 1. \end{aligned} \tag{4.14}$$

But, we claim that (4.14) is impossible for any $m, n, \partial q_1, \partial q_2, \partial p_1$, and ∂p_2 with $n \ge \partial q_1 \ge 0, n \ge \partial q_2 \ge 0, m \ge \partial p_1$, and $m \ge \partial p_2$. To see this, suppose that $\max(\partial p_1 + \partial q_2 ; \partial p_2 + \partial q_1) = \partial p_1 + \partial q_2$. In this case, (4.14) becomes

$$m + n + 2 - \min(n - \partial q_2 ; m - \partial p_2) \le \partial p_1 + \partial q_2 + 1,$$

or equivalently

$$\{m - \partial p_1\} + \{(n - \partial q_2) - \min(n - \partial q_2 ; m - \partial p_2)\} \le -1, \tag{4.15}$$

which is impossible as each term in braces on the left side of (4.15) is nonnegative. A similar argument gives a contraction if it is assumed that

$$\max\{\partial p_1 + \partial q_2 + \partial q_1\} = \partial p_2 + \partial q_1.$$

Therefore, it follows that $\partial q > \partial p$.

With $\partial q > \partial p$, (4.12) implies from (4.13) that

$$L := m + n + 2 - \min(n - \partial q_2 ; m - \partial p_2) \le \partial q = \partial q_1 + \partial q_2,$$

or

$$\{(n - \partial q_1)\} + \{(n - \partial q_2) - \min(n - \partial q_2 ; m - \partial p_2)\} \le n - (m+2). \tag{4.16}$$

Because each term in braces on the left side of (4.16) is nonnegative, we conclude that $0 \le n-(m+2)$, which establishes (4.9).

Now, assume $S \le 1/3$. Then $d \ge 2$ from (4.7), and (4.8) and Theorem 4 combine to give

$$L \le \partial p + 1 \quad \text{if } \partial p \ge \partial q, \tag{4.17}$$

and

$$L \le \left\lfloor \frac{\partial p + \partial q + 1}{2} \right\rfloor \quad \text{if } \partial q > \partial p. \tag{4.18}$$

Arguing as above, it similarly follows that assuming $\partial p \ge \partial q$ leads to a contradiction. This leaves only the possibility that $\partial q > \partial p$. Using (4.18), we then have

$$L := m+n+2-\min(m-\partial p_2; n-\partial q_2) \le \left\lfloor \frac{\partial p + \partial q + 1}{2} \right\rfloor. \tag{4.19}$$

Inequality (4.19) then implies

$$2m+2n+4-2\min(m-\partial p_2; n-\partial q_2) \le \partial p + \partial q + 1,$$

and on using (4.13), we have that

$$\begin{aligned} &2m+2n+4-2\min(n-\partial q_2; m-\partial p_2) \\ &\quad \le \max(\partial p_1+\partial q_2; \partial p_2+\partial q_1)+\partial q_1+\partial q_2+1. \end{aligned} \tag{4.20}$$

If $\max(\partial p_1+\partial q_2; \partial p_2+\partial q_1) = \partial p_1+\partial q_2$, then (4.20) may be rewritten as

$$\begin{aligned} &\{m-\partial p_1\} + 2\{(n-\partial q_2)-\min(n-\partial q_2; m-\partial p_2)\} \\ &\quad \le \partial q_1-(m+3) \le n-(m+3). \end{aligned} \tag{4.21}$$

But, as each term in braces on the left side of (4.21) is nonnegative, it is clear that (4.10) holds in this case. A similar argument establishes (4.10) when it is assumed that $\max(\partial p_1+\partial q_2; \partial p_2+\partial q_1) = \partial p_2+\partial q_1$.

To complete the proof of Theorem 5, we see that (4.1) implies

$$\gamma_{m,n} \ge 1/2 \quad \text{if } m+1 \ge n \ge 1, \tag{4.22}$$

while the reverse inequality holds from (2.18). Thus, we have

$$\gamma_{m,n} = 1/2 \quad \text{if } m+1 \ge n \ge 1,$$

the desired result of (4.3). Similarly, (4.2) implies

$$\gamma_{m,m+2} \ge 1/3 \quad \text{for any } m \ge 0, \tag{4.23}$$

while the reverse inequality holds from (2.18). Thus, we have

$$\gamma_{m,m+2} = 1/3 \quad \text{for any } m \ge 0,$$

the desired result of (4.4). □

Remark. We note that Trefethen and Gutknecht conjectured in [5] that $\gamma_{m,n}$ could only be zero if $m \le n+3$. Theorem 5 thus establishes the validity of their conjecture!

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